Art. XIX.—*On an Expeditious Practical Method of Harmonic Analysis.*[1]

By THOMAS R. LYLE, M.A.,

Professor of Natural Philosophy in the University of Melbourne.

(With Plates XXXI.-XXXIII.).

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1. Fourier has shown that if any function $f(t)$ $(=y$ say) of a variable t be such that

$$f(t)=f(t+\tau)=f(t+2\tau)=\text{etc.},$$

where τ is a constant, that is, if $f(t)$ be periodic in t, of period τ, then $f(t)$ can be expressed as the sum of a constant and a series of terms called harmonics, each of the form

$$a_p \sin p(\omega t-\theta_p),$$

where p has the values 1, 2, 3, 4, etc.,

and $\omega=2\pi/\tau$.

The number p is called the order of the harmonic, a_p its amplitude, and θ_p its phase.

If, in addition, $f(t)$ be such that

$$f(t)=-f(t+\tau/2),$$

then it is easy to see, by substituting $t+\tau/2$ for t, i.e., $\omega t+\pi$ for ωt in

$$y=a_0+\Sigma a_p \sin p(\omega t-\theta_p),$$

that in order for y_t to be $=-y_{t+\tau/2}$

$$a_0=0,\ a_2=0,\ a_4=0,\ \text{etc.}$$

Hence in this case the constant term vanishes and the harmonics, of which $f(t)$ is the sum, are all of odd order. When such is the case $f(t)$ is called an odd periodic function. This is the type generally met with in alternating electric current investigations.

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1 Appendix to the paper: "Preliminary Account of a Wave Tracer and Analyzer." Phil. Mag., Nov., 1903.

2. If we define the nth component (C_n say) of a periodic function $f(t)$ of period τ as the periodic function which is the sum of those harmonics of $f(t)$ whose orders are n, $3n$, $5n$, $7n$, etc., then

$$2nC_n = f(t) - f\left(t+\frac{\tau}{2n}\right) + f\left(t+2\frac{\tau}{2n}\right) - \quad . \quad . \quad . \quad .$$
$$- f\left(t+\overline{2n-1}\frac{\tau}{2n}\right). \qquad \text{(I.)}$$

For if we represent the expression on the right of the above equation by $\psi(t)$, we find by substituting successively for t, $t+\tau/2n$ and $t+\tau/n$ in it, that

$$\psi(t) = -\psi\left(t+\frac{\tau}{2n}\right) = \psi\left(t+\frac{\tau}{n}\right).$$

Hence $\psi(t)$ is an odd periodic function of period τ/n, that is to say, if

$$f(t) = a_0 + \Sigma a_p \sin p(\omega t - \theta_p),$$

where $p = 1, 2, 3, 4$, etc.,

then $\psi(t)$ is of the form

$$\psi(t) = \Sigma b_q \sin qn(\omega t - \beta_q),$$

where $q = 1, 3, 5, 7, 9$, etc.

In evaluating $\psi(t)$ therefore, only those harmonics whose arguments are $n\omega t$, $3n\omega t$, $5n\omega t$, etc., need be considered. Neglecting all other harmonics in the different f functions that make up $\psi(t)$, we find that the remainders in the $2n$ terms

$$f(t), \quad -f\left(t+\frac{\tau}{2n}\right), \quad f\left(t+2\frac{\tau}{2n}\right), \text{ etc.,}$$

are all equal, and that each remainder is the nth component of $f(t)$, hence

$$\psi(t) = 2nC_n.$$

3. If $f(t)$ itself contain only odd harmonics as in the case of alternate current periodic functions, then

$$f(t) = -f\left(t+\frac{\tau}{2}\right),$$

and equation I., §2, reduces to

$$nC_n = f(t) - f\left(t+\frac{\tau}{2n}\right) + \quad . \quad . \quad . \quad + f\left(t+\overline{n-1}\frac{\tau}{2n}\right). \qquad \text{(II.)}$$

The operation on $f(t)$ mathematically represented on the right hand side of equations I. or II., is practically performed on alternate current waves by the wave tracer and analyzer[1] designed by the author. In the simplest case, when $n = 1$, the wave tracer gives the first component of the periodic quantity operated on, which in the case of alternating electric currents is the full wave. By the movement of two pairs of brushes n can be made 3, or 5, or 7, in which cases the analyzer will give the 3rd, 5th, or 7th components of the wave respectively.

Now, in practical investigations with this apparatus on alternating current waves whose harmonic expressions were required, it was found much better to obtain by its means only the full wave trace, and then by an arithmetical process identical with the action of the analyzer and indicated by equation II. above, to obtain the 3rd and higher components of the wave, and thence to deduce its harmonics.

This method of harmonic analysis was drawn attention to in the paper already quoted, and though based on a different formula to that of Wedmore,[2] is practically similar to his. It is more suitable, however, for waves containing only odd harmonics, and as I have had considerable experience in its use during the last two years and have found it both expeditious and accurate, it is possible that a short account may be of value to those interested in alternating current work.

4. In wave graphs it is more convenient to use angular abscissae x where

$$x = \omega t = 2\pi t/\tau.$$

Making this substitution in the equation $y = f(t)$ it becomes $y = g(x)$ say, where

$$g(x) = g(x + 2\pi),$$

and if $f(t)$ is an odd periodic function as in the case of alternate current waves which we are now considering,

$$g(x) = -g(x + \pi) = g(x + 2\pi).$$

Substituting $g(x)$ for $f(t)$ in equation II. it becomes

$$n\mathrm{C}_n = g(x) - g(x + \pi/n) + g(x + 2\,\pi/n) - \quad . \quad . \quad . \\ + g(x + \overline{n-1}\,\pi/n),$$

from which we conclude that, if

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1 Lyle: "Preliminary Account of a Wave Tracer and Analyzer." Phil. Mag., Nov. 1903.

2 Wedmore: Journal Inst. Elect. Engineers, vol. xxv., p. 224 (1896).

$y_0, y_1, y_2, \text{——} y_{n-1}$ be n equi-spaced ordinates that exactly include half the wave, i.e., ordinates corresponding to the abscissae $x, x+\pi/n, x+2\pi/n, \quad . \quad . \quad . \quad . \quad x+\overline{n-1}\,\pi/n$ respectively, and called e.s. ordinates in the sequel; and if $N_0, N_1, N_2 \text{——} N_{n-1}$, be the ordinates of the nth component C_n whose abscissae are the same as those of $y_0, y_1, y_2, \text{——} y_{n-1}$ respectively, then

$$y_0 - y_1 + y_2 - \quad . \quad . \quad + y_{n-1} = nN_0 = -nN_1 = nN_2 = \quad = nN_{n-1}$$

when n is an odd number, and

$$y_0 - y_1 + y_2 - \quad . \quad . \quad . \quad - y_{n-1} = 0$$

when n is an even number, as we are now considering odd periodic functions only.

Thus from n e.s. ordinates of the original half wave we obtain only one ordinate per half wave of C_n, so that in order to obtain m e.s. ordinates per half wave of C_n it is necessary to have mn e.s. ordinates of the original half wave.

For instance, to obtain 3 e.s. ordinates of C_n we must measure $3n$ e.s. ordinates of $g(x)$. Let these be

$$y_0, y_1, y_2\ y_3, \text{——} y_{3n-1},$$

and let the corresponding ordinates of C_n be

$$N_0, N_1, N_2, N_3, \text{——} N_{3n-1},$$

then

$$y_0 - y_3 + y_6 - \quad . \quad . \quad + y_{3n-3} = nN_0 = -nN_3 = nN_6 = \quad = nN_{3n-3}$$
$$y_1 - y_4 + y_7 - \quad . \quad . \quad + y_{3n-2} = nN_1 = -nN_4 = nN_7 = \quad = nN_{3n-2}$$
$$y_2 - y_5 + y_8 - \quad . \quad . \quad + y_{3n-1} = nN_2 = -nN_5 = nN_8 = \quad = nN_{3n-1}$$

Subtracting now the ordinates of C_n so obtained from the corresponding y ordinates, we obtain a new set of $3n$ e.s. ordinates which are those of the original half wave with its nth component removed.

5. In practice it will generally be sufficient to determine the 1st, 3rd, 5th, 7th and 9th harmonics ($H_1\ H_3\ H_5\ H_7\ H_9$ say). This can be done with considerable accuracy when 15 e.s. ordinates of the original half wave are given.

Thus if these be

$$y_0, y_1, y_2, \quad . \quad . \quad . \quad y_{14}$$

corresponding to the angular abscissae

$$x_0, x_1, x_2, \quad . \quad . \quad . \quad x_{14}$$

where $x_1 - x_0 = x_2 - x_1 = \quad . \quad . \quad . \quad = x_{14} - x_{13} = \pi/15$,

and if z_0, z_1, z_2, z_3, z_4, be 5 e.s. ordinates of the half wave of

C_3, then

$$3z_0 = y_0 - y_5 + y_{10} = -3z_5 = 3z_{10}$$
$$3z_1 = y_1 - y_6 + y_{11} = -3z_6 = 3z_{11}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$3z_4 = y_4 - y_9 + y_{14} = -3z_9 = 3z_{14},$$

and if u_0, u_1, u_2 be 3 e.s. ordinàtes of the half wave of C_5, then

$$5u_0 = y_0 - y_3 + y_6 - y_9 + y_{12} = -5u_3 = 5u_6 = -5u_9 = 5u_{12}$$
$$5u_1 = y_1 - y_4 + y_7 - y_{10} + y_{13} = -5u_4 = 5u_7 = -5u_{10} = 5u_{13}$$
$$5u_2 = y_2 - y_5 + y_8 - y_{11} + y_{14} = -5u_5 = 5u_8 = -5u_{11} = 5u_{14}$$

the figure subscribed to each ordinate indicating the abscissa to which it corresponds.

Now the full wave

$$C_1 = H_1 + H_3 + H_5 + H_7 + H_9 + \text{etc.}$$

and $C_3 = H_3 + H_9 + H_{15}$

$$C_5 = H_5 + H_{15},$$

so that if H_{15} be neglected, and the sums of the corresponding ordinates of C_3 and C_5 be subtracted fram those of C_1, the fifteen remainders are ordinates of

$$H_1 + H_7 +$$

i.e., of H_1, if we neglect H_7.

If H_{15} cannot be neglected it can at once be removed from C_5 before subtracting from C_1, for as it is (*q.p.*) the 3rd component of C_5, of which we have 3 e.s. ordinates u_0, u_1, u_2, its three corresponding ordinates are i_0, $-i_0$, i_0 where $3i_0 = u_0 - u_1 + u_2$, hence H_5 will de completely given by

c_0, c_1, c_2 where

$$c_0 = u_0 - i_0, \quad c_1 = u_1 + i_0, \quad c_2 = u_2 - i_0.$$

H_{15} can now be taken from C_3, thus

$$z_0 - i_0, \quad z_1 + i_0, \quad z_2 - i_0, \quad z_3 + i_0, \quad z_4 - i_0$$

are the 5 e.s. ordinates of $H_3 + H_9$.

In order to determine H_3 and H_9 it will now be necessary to plot the 5 ordinates of $H_3 + H_9$, measure off 6 e.s ordinates from the smooth curve drawn through them, and from these determine their first component, that is 2 e.s. ordinates of H_9. These will completely determine H_9 if H_{27} etc., be neglected, and by subtracting them from the corresponding ordinates of $H_3 + H_9$ 6 e.s. ordinates of H_3 are obtained.

If H_7 cannot be neglected it will be necessary (if the original wave trace is not available) to plot the 15 ordinates of

H_1+H_7 obtained above, and from the smooth curve drawn through them to measure off 14 e.s. ordinates. From these, 2 e.s. ordinates of the half wave of H_7, and which determine H_7, can be obtained. By subtracting these from the corresponding ones of H_1+H_7, 14 corrected ordinates of H_1 are obtained.

6. It now remains to determine the amplitudes and phases of the harmonics of C_1 from their ordinates which we have obtained. It is easy to show that

$$\frac{2}{n}\left\{\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{n}\right)+\sin^2\left(\theta+\frac{2\pi}{n}\right)+\quad . \quad . \right.$$
$$\left. +\sin^2\left(\theta+\overline{n-1}\frac{\pi}{n}\right)\right\}=1,$$

from which we conclude that the square root of twice the mean of the squares of n e.s. ordinates of half a sine wave is equal to its amplitude.

Hence, with the help of a table of squares or of the quarter squares given in most sets of tables the amplitudes of H_1, H_3, etc., can be quickly determined.

[The rule that the amplitude is equal to $\pi/2\times$mean of the ordinates is only sufficiently accurate when a large number of ordinates is taken.]

If a_0, a_1, a_2, . . . a_{14} be the ordinates we have found for H_1 corresponding to the angular abscissae x_0, x_1, x_2 . . . x_{14} respectively, and if h_1, a be the amplitude and phase of H_1 or in other words, if

$$H_1=h_1\sin(\omega t-a),$$

then any of the equations

$$\sin(x_0-a)=a_0/h_1$$
$$\sin(x_1-a)=a_1/h_1$$
$$\sin(x_2-a)=a_2/h_1 \text{ etc.},$$

would determine a, provided the ordinates a_0, a_1, a_2, etc., are exactly those of a sine wave.

In practice, however, small upper harmonics will invariably be left in a_0, a_1, a_2, etc. [it may not have been thought worth while to remove H_7], and though their amplitudes may be negligably small, yet they might cause considerable error in the value of a when determined from only one of the above equations. Hence it is advisable to obtain four values of a from the first four ordinates on the rising side of the wave and four from the last four ordinates on the falling side, and take the mean of the eight. In this way we can to a great extent eliminate any error that might arise due to a harmonic even as low as the seventh not having been removed.

In a similar way the phases of H_3, H_5 etc., can be determined, but it must be remembered that if, for instance,

$$H_3 = h_3 \sin 3(\omega t - \beta),$$

and if $b_0, b_1, \ . \ . \ b_4$ are the ordinates of H_3 corresponding to the abscissae $x_0, x_1, \ . \ . \ x_4$, then

$$\sin 3(x_0 - \beta) = b_0/h_3 \text{ etc.}$$

similarly, if

$$H_5 = h_5 \sin 5(\omega t - \gamma)$$

with ordinates $c_0, c_1, c_2,$

then $\sin 5(x_0 - \gamma) = c_0/h_5$.

7. The wave to be analyzed may be given in either of two ways. We may have the complete trace of it obtained by the author's wave tracer by the photographic method, or by any form of oscillograph that gives a trace of the wave form ; or we may have the values of a definite number only of ordinates per half wave, such as would be obtained by the author's wave tracer by the galvanometer and scale method.

From the wave trace the complete harmonic expression can theoretically be obtained, but the impossibility of accurately measuring on the photograph, without elaborate apparatus, the different ordinates required leads to great inaccuracy in the result.

From a given number of e.s. ordinates only an approximate analysis can be obtained, more approximate, of course, as the number of ordinates is greater. When, however, each individual ordinate has been obtained with the accuracy of which the galvanometer and scale method is susceptible, the analysis obtained from fifteen such ordinates is much more reliable, as far as the harmonics up to the 9th are concerned, than that determined from any photographic trace.

I will therefore illustrate the method by applying it in full detail to the analysis of the wave whose 15 e.s. ordinates are

Table 1.

Abscissae (ωt)	12°	24°	36°	48°	60°	72°	84°	96°	108°	120°	132°	144°	156°	168°	180°	192°	204°
$H_1 + H_7 + H_{11}$	35	250	435	597	752	881	955	976	968	933	847	706	542	372	182	−35	−250
$-H_5$	43	4	−39	−43	−4	39	43	4	−39	−43	−4	39	43	4	−39		
$-C_3$	−73	35	135	182	153	73	−35	−135	−182	−153	−73	35	135	182	153		
Given ord^s - $f(t)$	65	211	339	458	603	769	947	1107	1189	1129	924	632	364	186	68		
	924	632	364	186	68												
	989	843	703	644	671												
	769	947	1107	1189	1129												
$3C_3$	220	−104	−404	−545	−458												
$3C_{15}$	1	−1	1	−1	1												
$3(H_3 + H_9)$	219	−103	−405	−544	−459												
	65	211	339	458	603	769											
	947	1107	1189	1129	924	632											
	364	186	68														
	1376	1504	1596	1587	1527	1401											
	1587	1527	1401														
$5C_5$	−211	−23	195														
$5C_{15}$	2	−2	2	$= \frac{1}{3}[-211+23+195]$													
$5H_5$	−213	−21	193														

Amp. H_1	Phase $H_1 = \alpha$: Sines	angles	$12° - \alpha$
306	·0355	2°. 2'	2°. 2'
15 625	·2533	14°. 40'	2° 40
47 306	·4407	26°. 9'	2° 9
89 102	·6048	37°. 13'	1 13
141 376			
194 040			
228 006			
238 144			
234 256	·7152	45°. 40'	2 20
217 622	·5491	33° 18'	2 42
179 352	·3768	22° 8'	1 52
124 609	·1854	10° 41'	1 19
73441			16. 17
34596		mean =	2°. 2'
8281			
1826062			
2			
15)3652124	$\alpha = 9°, 58'$.		
243475			
$h_1 = 987$			

Amp. H_3	Phase H_3: Sines	angles	$3(24° - \beta)$
11 990	·1907	11°. 0'	11°. 0'
2 652	·7500	48°. 35	12. 35
41 006			
73 984	·8500	58. 12	13. 48
52 670	·4056	23 55	12. 5
2× 182 302			49. 28
5)364604		mean	12 22
72921			
$3h_3 = 540$	$24° - \beta = 4°\text{-}7'$		
$h_3 = 180$ neg	$\beta = 19°\text{-}53'$ crossing down		

Amp. H_5	Phase H_5: Sines	angles	$5(36° - \gamma)$
11 342	·8212	55° 12'	55°. 12'
110			
9312			
20764	·0894	5°. 8'	54°. 52'
2			
3)41528		mean =	55°
13843	$36° - \gamma = 11°$		
$5h_5 = 235$			
$h_5 = 47$	$\gamma = 25°$ crossing up		

$$f(t) = 987 \sin(\omega t - 9°58') - 180 \sin 3(\omega t - 19°53') + 47 \sin 5(\omega t - 25°).$$

given in row 5 of Table I. Every figure necessary in the calculation will be given.

The first row of figures in Table I. are the abscissae x_0, x_1, etc. to which the given ordinates correspond. Space for three rows of figures is left, and then the 15 given ordinates are written down. These are divided into three sets of five each, and the numbers of the middle set are subtracted in order from the sums of first and last set, giving five numbers which are the corresponding ordidates of $3C_3$. Space for two or more rows is left, and the given ordinates are now written down as in the table, in two rows of six each and one row of three, in order. The columns formed are added and the last three of the sums are subtracted from the first three, giving three ordinates of $5C_5$. The first of these minus the second, plus the third, gives one ordinate of $15C_{15}$, whose other ordinates are got by alternating the sign. Subtracting $5C_{15}$ from $5C_5$ we obtain $5H_5$. Having obtained C_{15} we now subtract $3C_{15}$ from $3C_3$ and obtain $3(H_3 + H_9)$.

Above the given ordinates write those of C_3 with signs changed (row 4), and above these write those of H_5 with signs changed (row 3). Add rows 3, 4 and 5 to get row 2, in which are the ordinates of $H_1 + H_7 + H_{11}$ etc. Neglecting H_7, H_{11}, etc., as is done the analysis in Table I., we may consider the figures in row 2 as the ordinates of H_1, and neglecting H_9 we may consider the figures in row 11 as the ordinates of $3H_3$.

The first 15 numbers under Amp. H_1 are the quarter squares of the ordinates of H_1. Twice the sum of these is divided by 15, the number of ordinates, and the quotient is found to be the quarter square of 987. Hence h_1, the amplitude of H_1, is 987. Similarly for the amplitudes of H_3 and H_5.

Under the heading "phase of H_1," in the first column under sines, are the quotients got by dividing the first four ordinates on the rising side of H_1 and the last four on the falling side of H_1 by h_1; in the second column under angles are the corresponding angles, and in the third column are the eight values of $12°—\alpha$ deduced. The mean of these $2°\ 2'$ when subtracted from $12°$ gives the crossing point or phase of H_1 as $9°\ 58'$. Similarly for the phases of H_3 and H_5. It will be noticed that at the crossing point determined for H_3, H_3 crosses down, which is expressed analytically by writing its amplitude negative.

8. It will be noticed in the determination of the phase of H_1 in Table I., that the eight values of $12°-\alpha$ differ considerably from each other, indicating the presence in what we there take for H_1 of a considerable upper harmonic, probably H_7. In order to determine H_7 fourteen e.s. ordinates of the half wave are required. If the wave trace were given these could be measured off from it, but if, as in the case we are considering, only 15 original ordinates are given, it is necessary to plot the 15 ordinates of H_1+H_7 obtained in Table I., and from the smooth curve drawn through them to measure off 14 e.s. ordinates. This has been done and the values obtained are given in row 4, Table II., as well as the calculation necessary for the determination of H_7 and its elimination from H_1+H_7.

What is called the amplitude of H_1 in Table I. is really $\sqrt{2}$ R.M.S. (H_1+H_7), To get amp. H_1 it is better to remove the effect of H_7 by treating it as a correction, thus avoiding error that might be introduced in the plotting. This is easily done, since

$$\text{M.S}(H_1+H_7)=\frac{h_1^2}{2}+\frac{h_7^2}{2}$$

hence h_1 (corrected) $=\sqrt{\text{Amp.}(H_1+H_7)^2-h_7^2}$

$$=\sqrt{h_1^2\,(\text{uncorrected})-h_7^2}.$$

In Table II. the corrected crossing point of H_1 is determined, and it is seen to differ in phase only by 2 minutes from the value obtained in Table I.

The difference between the four values of $3(24°-\beta)$ when determining the crossing point of H_3 in Table I. point to the presence of a ninth harmonic, which exists as a third component in H_3+H_9. H_9 can, if desired, be determined by plotting the 5 ordinates obtained in Table I., measuring off from the curve six e.s. ordinates and proceeding as before. It will be found that

$$H_9=3\sin 9(\omega t-13°).$$

9. In Table III. is given most of the work required for the determination of the first six harmonics of a complete wave that contains harmonics both of odd and even orders. Twenty-four e.s. ordinates of the full wave are taken. This number is specially suitable, as it enables us to determine directly C_1, C_2, C_3, C_4

Table II.

Abscissae (ωt)	12°	24·9°	37·7°	50·6°	63·4°	76·3°	89·1°	102°	114·9°	127·7°	140·6°	153·4°	166·3°	179·1°	192°
H_1	34	253	459	641	793	905	969	987	951	871	749	587	397	186	−34
$-H_7$	−1	−11	1	11	−1	−11	1	11	−1	−11	1	11	−1	−11	
$H_1 + H_7$	35	264	458	630	794	916	968	976	952	882	748	576	398	197	
	794	916	968	976											
	952	882	748	576											
	398	197													
	2179	2259	2174	2182											
	2174	2182													
$7H_7$	5	77													

$$\text{Amp.}\,H_7 = \tfrac{1}{7}\sqrt{5^2+77^2} = 11$$

$$\text{Phase}\,H_7 = \delta$$

$$\sin 7(12^\circ-\delta) = 5/77$$

$$7(12^\circ-\delta) = 3^\circ.43'$$

$$\delta = 11^\circ.28'$$

Corrected Amp. H_1

$$= \sqrt{987^2 - 11^2} = 987$$

Corrected Phase H_1

sines	angles	$12^\circ - \alpha$
·0344	1° 58′	1° 58′
·2563	14° 51′	2°. 0′
·4650	27° 43′	2°. 0′
·6494	40° 30′	1° 58
·7588	49° 21′	2° 4′
·5947	36°.30′	2°. 4′
·4022	23°.43	2°. 0
·1884	10°.51′	2° 0

mean = 2°.

$\alpha = 10^\circ.$

$$f(t) = 987\sin(\omega t - 10^\circ) - 180\sin 3(\omega t - 19^\circ 53') + 47\sin 5(\omega t - 25^\circ) + 11\sin 7(\omega t - 11^\circ 28')$$

Table III.

Abscissae (ωt)	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
Given ords $f(t)$	101	280	479	609	634	602	535	399	213	77	33	−4	−88	−153	−159	−202	−377	−606	−720	−659	−495	−313	−156	−29
	88	153	159	202	377	606	720	659	495	313	156	29												
$2(H_1+H_3+H_5+\ )$	189	433	638	811	1011	1208	1255	1058	708	390	189	25												
$2(H_1+H_2+H_3+H_4+)$	202	560	958	1218	1268	1204	1070	798	426	154	66	−8												
$2(H_2+H_4+H_6)$	13	127	320	407	257	−4	−185	−260	−282	−236	−123	−33												
	185	260	282	236	123	33																		
$4(H_2+H_6+\ )$	198	387	602	643	380	29																		
$4(H_2+H_4+H_6)$	26	254	640	814	514	−8																		
$4H_4$	−172	−135	38	→			$H_4 = 45 \sin 4(\omega t - 42°)$																	
$4(H_2+H_6)$	198	387	602	643	380	29																		
	380	29																						
	578	416																						
	602	643																						
$12H_6$	−24	−227	→				$H_6 = -19 \sin 6(\omega t - 14°)$																	
$4H_6$	−8	−76	8	76	−8	−76																		
$4H_2$	206	463	594	567	388	105	→ $H_2 = 151 \sin 2(\omega t - 5°)$																	
$2(H_1+H_3+H_5)$	189	433	638	811	1011	1208	1255	1058	708	390	189	25												
	708	390	189	25																				
	897	823	827	836																				
	1011	1208	1255	1058																				
$6H_3$	−114	−385	−428	−222	→		$H_3 = -74 \sin 3(\omega t - 10°)$																	
$2H_3$	−38	−128	−143	−74	38	128	143	74	−38	−128	−143	−74												
$2(H_1+H_5)$	227	561	781	885	973	1080	1112	984	746	518	332	99												

$H_1 + H_5$ [10 e.s. ords from plot of]

ωt	15°	33°	51°	69°	87°	105°	123°	141°	159°	177°
H_1+H_5	113	307	413	467	524	557	472	324	210	79
	524	557	472	324						
	210	79								
	847	943	885	791						
	885	791								
$5H_5$	−38	152	→		→ $H_5 = 31 \sin 5(\omega t - 18°)$					
H_5	−8	30	8	−30	−8	30	8	−30	−8	30
H_1	121	277	405	497	532	527	464	354	218	49

$$H_1 = 540 \sin(\omega t - 2°)$$

$$f(t) = 540 \sin(\omega t - 2°) + 151 \sin 2(\omega t - 5°) - 74 \sin 3(\omega t - 10°) + 45 \sin 4(\omega t - 42°) + 31 \sin 5(\omega t - 18°) - 19 \sin 6(\omega t - 14°)$$

and C_6. To determine C_5, replotting will have to be resorted to if the full wave trace be not available.

At the top of Table III. are written the 24 given ordinates under their corresponding abscissae. From these ordinates the constant term of $f(t)$ has been removed. This can be done by aid of the formula

$$f(t)+f(t+\tau/n)+f(t+2\,\tau/n)+\quad .\quad .\quad .\quad +f(t+\overline{n-1}\,\tau/n)$$
$$=n[a_0+a_n\sin n(\omega t-\theta_n)+a_{2n}\sin 2n(\omega t-\theta_{2n})$$
$$+a_{3n}\sin 3n(\omega t-\theta_{3n})+\text{etc.}] \qquad \text{(III.)}$$

which can be easily established by the method used in § 2.

From this formula we see that the mean of n e.s. ordinates embracing one period of a periodic function is equal to its constant term, if its nth, $2n$th, etc., harmonics are neglected.

Returning to Table III., we add the second twelve ordinates with their signs changed to the first twelve, in order, and obtain 12 e.s. ordinates of $2C_1$, i.e., of $2[H_1+H_3+H_5+\quad]$. (See equation I., § 2).

Subtracting these from twice the given ordinates, those of $2[H_2+H_4+H_6+\quad]$ are left, and the remainder of the work proceeds as in Table I.

$2[H_2+H_4+H_6+\text{etc.}]$ could be obtained directly from the 24 given ordinates by adding the second 12 to the first 12 of them, in order. (See formula III., § 9).

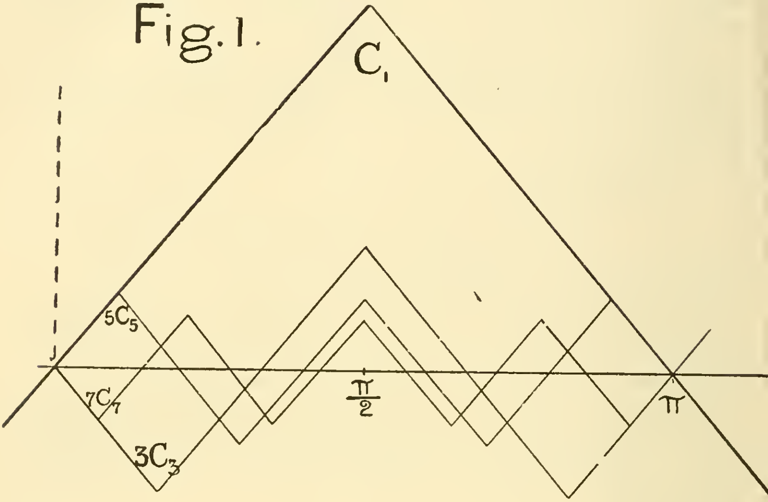
The amplitudes and phases of the different harmonics were determined as in Table I., but the figures necessary in their calculation are not given.

The following are interesting applications of the above method to more general harmonic analysis.

10. To obtain the harmonic expression for the odd periodic function whose graph for half a period is the sides of an isosceles triangle of altitude h. See Fig. 1.

Taking o and π as the abscissae of the extremities of the base, relative values of any number of e.s. ordinates can be written down, and any component at once obtained. Thus, 30 e.s. ordinates would be 0, 1, 2, 3, . . 14, 15, 14, . . 2, 1, and these correspond to an altitude 15.

Fig. 1.



It will be found that all the components (i.e. 3rd, 5th, etc., in this case) are the sides of isosceles triangles passing through the origin, and that the altitudes are

$-h/3^2$, $h/5^2$, $-h/7^2$, etc. respectively. (See Fig. 1.).

(The same can be quickly arrived at geometrically).

Hence, if the full wave or C_1 be represented by

$$C_1 = a_1 \sin(\omega t - \theta_1) + a_3 \sin 3(\omega t - \theta_3) + a_5 \sin 5(\omega t - \theta_5) + \text{etc.},$$

its third component C_3 is

$$= -\frac{1}{3^2}[a_1 \sin(3\omega t - \theta_1) + a_3 \sin 3(3\omega t - \theta_3) + a_5 \sin 5(3\omega t - \theta_5)$$
$$+ \text{etc.}],$$

and its fifth component C_5 is

$$\frac{1}{5^2}[a_1 \sin(5\omega t - \theta_1 + a_3 \sin 3(5\omega t - \theta_3) + a_5 \sin 5(5\omega t - \theta_5) + \text{etc.}]$$

and so on, but by definition C_3 and C_5 are also given by

$$C_3 = a_3 \sin 3(\omega t - \theta_3) + a_9 \sin 9(\omega t - \theta_9) +$$
$$C_5 = a_5 \sin 5(\omega t - \theta_5) + a_{15} \sin 15(\omega t - \theta_{15}) +$$

hence, identifying the expressions for the same components, we find that

$$a_1 = -3^2 a_3 = 5^2 a_5 = -7^2 a_7 = \text{etc.},$$
$$\theta_1 = 3\theta_3 = 5\theta_5 = 7\theta_7 = \text{etc.},$$